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Abstract

In the framework of the QCD string approach it is shown that the spin-averaged masses $\bar{M}(nL)$ of all low-lying light mesons are well described using the string tension σ as the only parameter. The Regge slope α'_L and the intercept $\alpha_L(0)$ of the Regge L -trajectory for $\bar{M}(nL)$ are calculated analytically and turn out to be $\alpha'_L = 0.80 \text{ GeV}^{-2}$ (for $L \leq 4$) and $\alpha_L(0) = -0.34$, in good agreement with the experimental data: $\alpha'_{L\text{exp}} = 0.81 \pm 0.01 \text{ GeV}^{-2}$, $\alpha_{L\text{exp}}(0) = -0.30 \pm 0.02$. To obtain this strong agreement with the data the nonperturbative quark self-energy contributions to the meson masses must be taken into account, which appeared to be large and negative for small values of L , and are even for larger values of L important for a close fit. From the present analysis of the meson spectra the restriction $\alpha_s \leq 0.40$ on the strong coupling is required.

PACS

I. INTRODUCTION

The spectra of hadrons form an extremely important test ground for nonperturbative QCD. The scaling property of QCD tells us that in the end all characteristics of hadrons must depend on a single parameter, say σ or Λ_{QCD} . Till now, all attempts to estimate hadronic spectra with an accuracy comparable to the uncertainties in the experimental data, have relied on models that contain several, in some cases even many, parameters. In sharp contrast to that, the formalism adopted here that uses the QCD-string Hamiltonian, relies on only one free parameter, the string constant σ , which must ultimately be related in a one-to-one correspondence with Λ_{QCD} , as it is intimately related to the confining character of QCD. The string tension can be extracted from experiment, in particular from the slope of the leading Regge trajectory, and in the present paper we use it to describe the spectra of light mesons.

The QCD string approach developed in recent years [1] - [3] starts from first principles in a very direct way, namely from the QCD Lagrangian. In Ref. [2] the relativistic Hamiltonian H_R for the light mesons with spinless quarks was derived with only one approximation—the string was taken to be a straight line. This approach can be applied to most of the light and

heavy-light mesons with the exception of the π - and K -mesons, for which spin and chiral effects are very important and cannot be considered as a perturbation.

With the use of this relativistic Hamiltonian many important features of the light meson spectra can be clarified, in particular the constituent mass of a quark $\mu(nL)$ is defined in a rigorous way (we use the current quark mass $m = 0$), and appears to be dependent on the quantum numbers of a given state [1]. This is not so in the most common potential models, in particular in the relativized potential model (RPM) where the constituent mass of a quark is actually a fitting parameter, e.g. in Ref. [4] $m = 220$ MeV, and kept fixed for states with different L and n [4], [5].

For the Hamiltonian H_R the correct Regge slope, equal to $1/(2\pi\sigma)$ was obtained for large L [6] while for $L \leq 4$ the string correction, Δ_{str} , is shown to modify the Regge slope by about 15% [2], [6].

In the RPM in order to fit the hadron masses a negative constant (a fitting parameter) is always added to the meson mass while in the QCD string Hamiltonian there is no fitting parameter at all. Instead, the nonperturbative quark self-energy contribution Δ_{SE} to the meson mass has been shown to be important and this term was analytically calculated in Ref. [7]. Δ_{SE} is negative and has rather a large magnitude (of the order of -400 to -300 MeV) for all states with $L \leq 5$ and gives rise to the correct value of the Regge intercept.

Here we consider in detail the spin-averaged meson masses, $\bar{M}(nL)$, or the centers of gravity of the nL -multiplets (i.e. neglect the hyperfine and fine-structure splittings) for which the physical picture is simpler and at the same time more universal since the parameters do not depend on spin and isospin.

We concentrate mostly on the orbital excitations with $n = 0$ for which experimental data exist for all ground states with $L \leq 5$. Then for the linear confining potential σr all meson masses $\bar{M}(nL)$ can be expressed through a single parameter—the string tension σ . The values of the slope and the intercept of the Regge L-trajectory ($L \leq 4$) will be calculated analytically: their numerical values are $\alpha'_L = 0.80 \text{ GeV}^{-2}$ ($\sigma = 0.18 \text{ GeV}^2$) and $\alpha_L(0) = -0.34$ turn out to be in very good agreement with the experimental numbers. From the Regge slope a restriction on the admissible values of the string tension follows: $\sigma = 0.18 \pm 0.005 \text{ GeV}^2$ for the pure linear potential and $\sigma = 0.19 \pm 0.01 \text{ GeV}^2$ if the Coulomb interaction is taken into account.

The Coulomb contribution is mainly important for the $1S$ and the $1P$ states having values in the range -200 to -100 MeV, and is considered here in a twofold way: from exact calculations with the linear plus Coulomb potential and also when the Coulomb interaction is considered as a perturbation; both considerations give very close results. For $\sigma = 0.19 \text{ GeV}^2$ the QCD coupling is $\alpha_s = 0.39$ which is typical for heavy quarkonia, and from our analysis of the meson spectra the following restriction on the strong coupling $\alpha_s \leq 0.42$, is obtained. This number is in a good agreement with the two-loop value of the freezing coupling constant obtained in background field theory [8].

II. EXPERIMENTAL DATA

The experimental numbers for the spin-averaged masses $\bar{M}(L)$, or the centers of gravity of the $1L$ -multiplet (the ground states with $n = 0$), are presented in Table I and need some remarks.

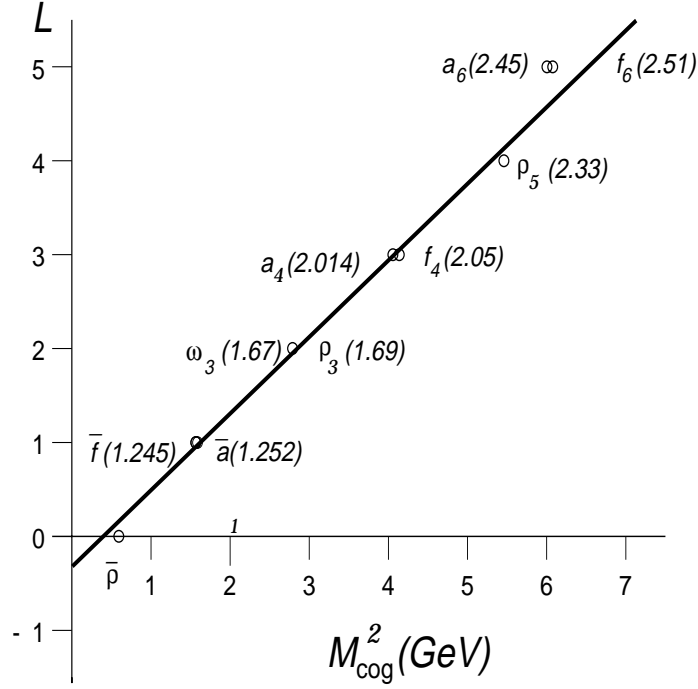


FIG. 1. The Regge L -trajectory for the light mesons.

First, all members of the 1^3P_J multiplet are supposed to be known: $a_2(1318)$, $a_1(1235)$, and the $a_0(980)$ too, are considered to form the 1^3P_0 multiplet with $\bar{M}(1P) = 1252$ MeV. Similarly, for the $f_J(1P)$ mesons the spin-averaged mass is 1245 MeV [10].

Second, in the case of $L = 2$, the fine-structure splittings of the $1D$ -wave mesons are supposed to be suppressed as compared to the P -wave states [11]. As a result, for all members of the $1D_J$ -multiplet, e.g. the $\rho_3(1.69)$ and $\pi_2(1.67)$, their masses are very close to each other and one can expect that the true value of $\bar{M}(1D)$ lies between these two values. The same would be valid for the isoscalar mesons, if they were not mixed with other hadronic states, and just this situation is observed in experiment where the masses of the $\omega_3(1.67)$ and the $\omega(1.65)$ have values close to the corresponding isovector mesons [10].

Due to the suppression of the matrix elements (m.e.) like $\langle 1/r^3 \rangle$ the spin splittings for the higher orbital excitations like $1F$, $1G$, etc. should be even smaller than for the $1D$ mesons. Therefore the masses of the $a_4(2.01)$ and the $f_4(2.03)$ are supposed to be close to $\bar{M}(1F)$ as well as the masses of the $\rho_5(2.30)$ with $L = 4$, and $a_6(2.45)$ and $f_6(2.47)$ with $L = 5$, lie close to their centers of gravity. The masses of all orbital excitations ($n = 0$) can be nicely described by the Regge L -trajectory (see Fig.1):

$$\bar{M}^2(L) = (1.23 \pm 0.02) L + 0.38 \pm 0.02 \text{ (GeV}^2\text{)}, \quad (2.1)$$

or

$$L = 0.81\bar{M}^2(L) - 0.30 \quad (2.2)$$

with the following Regge slope and intercept:

$$\alpha'_{L\text{exp}} = 0.81 \pm 0.01 \text{ (GeV}^2\text{)} \quad \text{and} \quad \alpha_{L\text{exp}}(0) = -0.30 \pm 0.02 \text{ (} L \leq 4\text{)}. \quad (2.3)$$

and both values have a small experimental error.

Note that for the leading ρ -trajectory:

$$J = \alpha'_J M^2(J) + 0.48 \quad (2.4)$$

the slope $\alpha'_J = 0.88 \text{ GeV}^{-2}$ and the intercept $\alpha_J(0) = 0.48$ are larger since their values depend on the spin contributions. On the contrary, the L -trajectory is a universal one and in the approximation of closed channels it is the same for isovector and isosinglet mesons.

In our paper the meson masses, Regge slope and Regge intercept will be calculated analytically in the framework of the QCD string approach.

III. RELATIVISTIC HAMILTONIAN

In Ref. [2] the relativistic Hamiltonian H_R was derived from the meson Green's function which was written in the Feynmann-Schwinger representation with the use of two auxiliary (einbein) fields, $\mu(\tau)$ and $\nu(\beta)$. For quarks with equal current mass m it has the following form :

$$\begin{aligned} H_R = & \frac{p_r^2 + m^2}{\mu(\tau)} + \mu(\tau) \\ & + \frac{\vec{L}^2}{r^2} \left[\mu(\tau) + 2 \int_0^1 d\beta \nu(\beta) \left(\beta - \frac{1}{2} \right)^2 \right]^{-1} \\ & + \frac{1}{2} \sigma^2 r^2 \int_0^1 \frac{d\beta}{\nu(\beta)} + \frac{1}{2} \int_0^1 d\beta \nu(\beta). \end{aligned} \quad (3.1)$$

The two auxiliary fields $\mu(\tau)$ and $\nu(\tau)$ in H_R are operators which depend on the proper time τ of dimension (length)², introduced by Schwinger [12]. By definition the field operator $\mu(\tau)$ is

$$\mu(\tau) = \frac{1}{2} \frac{dt}{d\tau}, \quad (3.2)$$

where t is the actual time.

In Eq. (3.1) m is the current quark mass, which for a light quark (antiquark) will be taken equal zero; \vec{L} is the angular orbital momentum, $\vec{L} = \vec{r} \times \vec{p}$, and the operator $p_r^2 = (\vec{p} \cdot \vec{r})^2 / (r^2)$. The constant σ determining the nonperturbative potential is the string tension.

In many cases it is convenient to rewrite H_R as a sum of two terms:

$$H_R = H_R^{(1)} + \Delta H_{\text{str}}, \quad (3.3)$$

with the “unperturbed” Hamiltonian $H_R^{(1)}$ defined by

$$H_R^{(1)} = \frac{\vec{p}^2 + m^2}{\mu(\tau)} + \mu(\tau) + \frac{1}{2} \sigma^2 r^2 \int_0^1 \frac{d\beta}{\nu(\beta)} + \frac{1}{2} \int_0^1 d\beta \nu(\beta). \quad (3.4)$$

where in $H_R^{(1)}$ we have included the term $L^2/(\mu r^2)$ and subtracted the same term to give the string correction ΔH_{str}

$$\begin{aligned}\Delta H_{\text{str}} &= -\frac{\vec{L}^2}{r^2} \left[\frac{1}{\mu(\tau)} - \frac{1}{\mu + 2 \int_0^1 d\beta \nu(\beta) (\beta - \frac{1}{2})^2} \right] \\ &= -\frac{\vec{L}^2}{\mu r^2} \frac{2 \int_0^1 d\beta \nu(\beta) (\beta - \frac{1}{2})^2}{\mu + 2 \int_0^1 d\beta \nu(\beta) (\beta - \frac{1}{2})^2}.\end{aligned}\quad (3.5)$$

If L is not large, then the term ΔH_{str} appears to be relatively small and can be considered as a correction to the Hamiltonian $H_R^{(1)}$ [6] but for large L the representation of H_R as the sum Eq. (3.3) is of no use, since in this case both terms are equally important. Note that to get the expression (3.4) one needs the following definition

$$\vec{p}^2 = p_r^2 + \frac{\vec{L}^2}{r^2}.\quad (3.6)$$

The simplest Hamiltonian H_0 with $L = 0$ is a special case of H_R (or $H_R^{(1)}$) with \vec{p}^2 replaced by p_r^2 .

IV. THE EXTREMAL VALUES OF THE OPERATORS μ AND ν

To understand the physical meaning of the auxiliary fields $\mu(\tau)$ and $\nu(\tau)$ let us find their extremal values. First, in the Hamiltonian H_R we determine the variable $\nu(\beta)$ from the extremum conditions

$$\frac{\delta H_R^{(1)}}{\delta \nu(\beta)} = 0, \quad \frac{\delta H_R^{(1)}}{\delta \mu(\tau)} = 0.\quad (4.1)$$

Then one finds that $\nu(\beta)$, which is an operator in general, does not depend on the string parameter β and is equal to

$$\nu_0(\beta) = \sigma r,\quad (4.2)$$

i.e. is actually the energy density along the string.

With the use of Eq. (4.2) the Hamiltonian $H_R^{(1)}$ reduces to a simpler operator:

$$H_R^{(1)} = \frac{\vec{p}^2 + m^2}{\mu(\tau)} + \mu(\tau) + \sigma r,\quad (4.3)$$

where $\mu(\tau)$ is still an operator in the Hamiltonian formalism. Its extremum can be found from the second extremum condition (4.1)

$$\mu(\tau) = \sqrt{\vec{p}^2 + m^2}, \text{ for } H_R^{(1)}, \quad \mu_0(\tau) = \sqrt{p_r^2 + m^2}, \text{ for } H_0,\quad (4.4)$$

i.e. the extremal value of μ is one half the kinetic energy operator. Substituting it to the Hamiltonians $H_R^{(1)}$ one obtains

$$H_R^{(1)} = 2\sqrt{\vec{p}^2 + m^2} + \sigma r \quad (4.5)$$

giving the rise to an eigenvalue equation that is identical to the spinless Salpeter equation (SSE) with a linear potential

$$H_R^{(1)}\psi(nL) = M_0(nL)\psi(nL). \quad (4.6)$$

This equation has been used in the RPM for many years [4], [5], however, in the RPM instead of the current mass m a fitting mass is usually used, e.g. $m = 220$ MeV in [5]. Nevertheless, due to our derivation of Eq. (4.6) the connection between the QCD string theory and the RPM is established.

V. THE CONSTITUENT QUARK MASS

Although the constituent mass μ is not explicitly present in $H_R^{(1)}$, it enters many important physical characteristics like the spin splittings and magnetic moments, and also in the string and self-energy corrections, therefore it must not be left in as an operator. The simplest way to solve this is to define μ as the expectation value of one half the quark kinetic energy operator Eq. (4.4), i.e.,

$$\mu_0(nL) = \langle \sqrt{\vec{p}^2 + m^2} \rangle_{nL}. \quad (5.1)$$

Note that the eigenvalues $M_0(nL)$ in Eq. (4.5) for the linear potential σr are connected with μ_0 as follows

$$M_0(nL) = 4\mu_0(nL). \quad (5.2)$$

The values of μ_0 can be expressed through a single parameter—the string tension σ and the universal numbers $a(nL)$ given by

$$\mu_0(nL) = \sqrt{\sigma} a(nL). \quad (5.3)$$

This relation is a manifestation of the scaling property of the SSE in the case $m = 0$.

Another definition of the constituent mass, denoted by $\tilde{\mu}_0$, was used in Refs. [1], [3] in the so called “einbein approximation” (EA) where the second extremum condition in Eq. (3.6) is written not for the operator $H_R^{(1)}$ but for the eigenvalues $M_0(nL)$. A priori it is not clear whether in both definitions the extremal values $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ coincide or not, therefore let us compare them. In the EA Eq. (4.6) is rewritten as

$$\left[\frac{\vec{p}^2 + m^2}{\tilde{\mu}(\tau)} + \sigma r \right] \tilde{\psi} = \varepsilon(nL) \tilde{\psi}, \quad (5.4)$$

with

$$\varepsilon(nL) = \left(\frac{\sigma^2}{\tilde{\mu}} \right)^{1/3} A(nL), \quad (5.5)$$

i.e., it reduces to the Airy equation with $\varepsilon(nL) = M_0(nL) - \tilde{\mu}$ and the quantities $A(nL)$ in Eq. (5.5) are the zeros of the Airy function. The constituent mass $\tilde{\mu}_0(nL)$ is now determined by the condition

$$\frac{d\varepsilon(\tilde{\mu})}{d\tilde{\mu}} + 1 = 0, \quad (m = 0). \quad (5.6)$$

Then from Eqs. (5.5, 5.6) one obtains that

$$\tilde{\mu}_0(nL) = \sqrt{\sigma} \left(\frac{1}{3} A(nL) \right)^{3/4} = \sqrt{\sigma} \tilde{a}(nL). \quad (5.7)$$

To compare $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ one can use the numbers presented in Appendix A (see Tables VII and VIII) from which the corresponding universal numbers $a(nL)$ and $\tilde{a}(nL)$ can be determined.

The largest difference between $\mu_0(nL)$ and $\tilde{\mu}_0(nL)$ was found for S waves and is increasing with growing radial quantum number n from 5% for the $1S$ state to 7% for the $5S$ state. However, this difference is falling with increasing L , being only 1.7% for $L = 5$ ($n = 0$). So, μ_0 and $\tilde{\mu}_0$ are numerically very close. In contrast to the eigenvalues $M_0(nL)$ for the Salpeter and Airy equations a large difference is found between some matrix elements (m.e.) like $\langle 1/r^3 \rangle$ (for any $L \neq 0$) which define the fine-structure splittings. This difference can be as large as 30-50% in some cases (see Tables VII, VIII). Moreover, while for the SSE these m.e. are growing, they are slightly decreasing for the Airy equation. It is worth to notice that these differences between the m.e. would be much larger if a fixed constituent mass, as in potential models, would be used.

The reason behind such discrepancies may be connected with the different asymptotic behavior of the wave functions(w.f.). For the SSE Eq. (4.5) it falls as $\exp(-\sqrt{\sigma} r)$ [13] while for the Airy Eq. (5.4) the w.f. decreases as $\exp(-\sqrt{\tilde{\mu}_0 \sigma} r^{3/2})$. Therefore the definition (4.6) of the constituent quark mass as well as the calculations of the m.e. with the use of the unperturbed Hamiltonian $H_R^{(1)}$ has to be considered as preferable compared to the EA.

Note a useful relation between the m.e.:

$$\langle \sigma r \rangle = 2\mu_0(nL) \quad (5.8)$$

and

$$\langle 1/r \rangle = \sqrt{\sigma} \langle 1/\rho \rangle_{nL}, \quad (5.9)$$

where $\langle 1/\rho \rangle$ is independent of σ but does depend on the quantum numbers.

Although the approximate string Hamiltonian $H_R^{(1)}$ coincides with the one used in the RPM, there are essential differences between them.

First, in Eq. (4.6) only the current mass m is present and for a light quark it is supposed to be equal to zero, while e.g. in Ref. [5] this mass was taken equal 220 MeV, being actually a fitting parameter.

Second, the constituent mass of a light quark is defined in a rigorous way as the average quark kinetic energy and appears to be different for states with different quantum numbers n and L .

TABLE I. The masses $M_0^2(L)$ and $M_0^2(\text{approx}) = 8L + 3\pi\sigma$ for the ground states ($n = 0$) with $L \leq 6$ ($\sigma = 0.18 \text{ GeV}^2$).

L	0	1	2	3	4	5	6
$M_0^2(nL)$	1.7940	3.2126	4.6431	6.0777	7.5142	8.9518	10.3900
$8L\sigma + 3\pi\sigma$	1.696	3.1365	4.5765	6.0165	7.4565	8.8965	10.3365
difference	5.4%	2.3%	1.4%	1.0%	0.77%	0.62%	0.5%

Third, the spin-dependent interactions will be defined by the same constituent mass $\mu_0(nL)$ Eq. (5.1) and therefore, to describe the spin structure of the light mesons no extra parameter besides the QCD strong coupling will be introduced.

Finally, in the RPM the string and self-energy corrections, which will be discussed in next Sections, are absent.

VI. THE STRING CORRECTION AND THE SLOPE OF THE REGGE TRAJECTORY

It is known that for the Salpeter equation (4.6) (or for the unperturbed Hamiltonian $H_R^{(1)}$) the squared masses $M_0^2(nL)$ can be approximated (with an accuracy of about 1% for $L \neq 0$) by the “string formula” [16],

$$M_0^2(\text{approx}) = 8\sigma L + 4\pi\sigma(n + 3/4). \quad (6.1)$$

The exact values of $M_0^2(nL)$ together with those of $M_0^2(\text{approx})$ ($L \leq 6$, $n = 0$) are given in Table I from which one can see that the differences between them are indeed $\leq 1\%$ for $L \geq 2$.

As is clear from the approximation (6.1), the slope of the Regge trajectory for the SSE is $(8\sigma)^{-1}$, i.e. $\alpha'_L = 0.69 \text{ GeV}^{-2}$ for $\sigma = 0.18 \text{ GeV}^2$, which is 17% smaller than the experimental number Eq. (2.3), $\alpha'_L = 0.81 \pm 0.01 \text{ GeV}^{-2}$. Note that the string corrections which come from the term Eq. (3.5) are also proportional to L and therefore affect the Regge slope. The situation appears to be different in two domains: $L \leq 4$ and $L \geq 5$ respectively, and we consider them separately.

A. Case A. $L \leq 4$

By the definition (3.5) ΔH_{str} gives a negative correction to the eigenvalues $M_0(nL)$; its magnitude turns out to be relatively small, $\sim -100 \text{ MeV}$, and therefore this term can be considered as a perturbation [6].

$$\Delta_{\text{str}}(nL) = \langle \Delta H_{\text{str}} \rangle = -\frac{\sigma L(L+1)}{\mu_0(nL)} \left\langle \frac{1}{r(6\mu_0 + \sigma r)} \right\rangle. \quad (6.2)$$

In Eq. (6.2) we have used that the integral $\int_0^1 d\beta (\beta - 1/2)^2$ is equal to $1/12$ and the operators ν and μ were replaced by their extremal values Eq. (4.2) and Eq. (4.4). The factor in brackets

TABLE II. The string corrections Δ_{str} in MeV and the mass $M_0(L)$ in GeV, for the ground states ($L \leq 6$).

L	1	2	3	4	5	6
$M_0(L)$	1.7924	2.1549	2.4653	2.7412	2.9920	3.2234
$\Delta_{\text{str}}(L)$	-52.9	-86.9	-113.0	-132.7	-153.7	-170.7
$\Delta_{\text{str}}(\text{asym})^a$	—	—	—	-142.4	-182.9	-219.1

^{a)} For $L \leq 3$ the asymptotic formula Eq. (6.12) is not applicable.

TABLE III. The squared masses $M^2 = (M_0 + \Delta_{\text{str}})^2$ in GeV^2 for the ground states ($L \leq 6$, $\sigma = 0.18 \text{ GeV}^2$).

L	1	2	3	4	5	6
$(M_0 + M_{\text{str}})^2$	3.026	4.277	5.533	6.794	8.0567	9.319
$M_{\text{str}}^2(\text{asym})^a$	—	—	—	6.754	7.891	9.026

^{a)} see footnote to Table II

can also be approximated (with an accuracy better than 3%) replacing σr by $\langle \sigma r \rangle$. Then the string correction is

$$\Delta_{\text{str}}(nL) = -\frac{\sigma L(L+1)\langle 1/r \rangle}{\mu_0(6\mu_0 + \langle \sigma r \rangle)}. \quad (6.3)$$

Due to the relations (5.2) and (5.8) for the linear potential the correction Δ_{str} becomes

$$\Delta_{\text{str}} = -\frac{\sigma L(L+1)\langle 1/r \rangle}{8\mu_0^2} = -\frac{2\sigma^{3/2}\langle 1/\rho \rangle L(L+1)}{M_0^2}. \quad (6.4)$$

Note that in Eq. (6.4) the m.e. $\langle 1/\rho \rangle \sqrt{L+1}$ is almost constant, varying from 0.787 for $L = 1$ to 0.741 for $L = 4$ (see Table VII). The values of Δ_{str} (using the m.e. $\langle 1/r \rangle$ from Tables VIII and IX) are given in Table II.

For comparison in Table II the string corrections valid for large L ($L \geq 5$) (see the asymptotic string correction formula Eq. (6.13)) are also given.

Now one can analytically calculate the Regge slope for the “corrected” mass:

$$M(nL) = M_0(nL) + \Delta_{\text{str}}(nL) \quad (L \leq 4), \quad (6.5)$$

then the squared mass

$$M^2(L) = M_0^2(L) - \frac{4\sigma^{3/2}L(L+1)\langle 1/\rho \rangle}{M_0} + \Delta_{\text{str}}^2. \quad (6.6)$$

If one neglects Δ_{str}^2 in Eq. (6.6) which is small ($\leq 0.016 \text{ GeV}^2$ for $L \leq 4$) and uses the approximation (6.1) for $M_0^2(L)$ then for the orbital excitations with $n = 0$ the squared mass Eq. (6.6) becomes

$$M^2(L) = 8\sigma L - \sigma \frac{\sqrt{2}\langle 1/\rho \rangle(L+1)}{\sqrt{L+3\pi/8}} + 3\pi\sigma = (\alpha'_L)^{-1} L + 3\pi\sigma \quad (6.7)$$

where the inverse Regge slope in Eq. (6.7) is

$$(\alpha'_L)^{-1} = \left(8 - \frac{\sqrt{2}\langle 1/\rho \rangle (L+1)}{\sqrt{L+3\pi/8}} \right) \sigma = (6.95 \pm 0.02)\sigma. \quad (6.8)$$

The values of $(\alpha'_L)^{-1}$ are practically constant, see the numbers in Table IX, varying from the value 6.930σ for $L=1$ to 6.970σ for $L=4$ and we take here $(\alpha'_L)^{-1} = (6.95 \pm 0.02)\sigma$. Then

$$M^2(L) = 6.95\sigma L + 3\pi\sigma \quad (6.9)$$

or

$$L = 0.144/\sigma M^2(L) - 1.358. \quad (6.10)$$

It gives for $\sigma = 0.18 \text{ GeV}^2$ the Regge slope

$$(\alpha'_L)^{-1} = 1.25 \text{ GeV}^2 \text{ or } \alpha'_L = 0.80 \text{ GeV}^{-2}, \quad (6.11)$$

in good agreement with the experimental number given in Eq. (2.3) $\alpha'_L(\text{exp}) = 0.81 \pm 0.01 \text{ GeV}^{-2}$. Thus, due to the string corrections we have obtained the correct Regge slope for the spin-averaged masses. However, the intercept in Eq. (6.10) has a very large magnitude and an additional contribution to the meson mass must be taken into account. We discuss this contribution in Sect. VII

B. Case B. Large L

For large L the extremal value of the operator ν is not equal to σr but turns out to depend on the parameter β as well as on the operator $\mu(\tau)$. In this case it is a difficult problem to find the exact eigenvalues $M(\text{asym})$ of the Hamiltonian H_R , therefore in Ref. [6] the eigenvalues of H_R have been calculated in the quasiclassical approximation with the following result,

$$M^2(\text{asym}) = 2\pi\sigma\sqrt{L(L+1)} + 3\pi\sigma. \quad (6.12)$$

Here, in the asymptotic mass formula (6.12) the string correction is already taken into account and the constant $3\pi\sigma$ is kept to match the solutions for large L to those for $L \leq 4$. Now, for comparison one can formally define the string correction for large L as the difference between the asymptotic mass Eq. (6.12) and the unperturbed mass $M_0(nL)$ Eq. (5.2)

$$\Delta_{\text{str}}(\text{asym}) = \sqrt{3\pi\sigma + 2\pi\sigma\sqrt{L(L+1)}} - M_0(L), \quad (L \gg 1, n=0). \quad (6.13)$$

The asymptotic masses are less than $M_0(L)$ Eq. (4.6) for $L \geq 4$. The magnitude of Δ_{str} is increasing with growing L and for $L=6$ $\Delta_{\text{str}}(\text{asym})$ is already $\approx 220 \text{ MeV}$.

From the numerical values of $\Delta_{\text{str}}(\text{asym})$ (see Table III) one can see that for $L=4$ both string corrections, from the asymptotic formula Eq. (6.13) and from Eq. (6.5), practically

coincide and in what follows the string correction will be taken in the form (6.5) for $L \leq 4$ and from Eq. (6.13) for $L \geq 5$ (when the masses $M(\text{asym})$ are smaller, see Table III).

For $L \gg 1$ the Regge slope in Eq. (6.13) is $(2\pi\sigma)^{-1}$, i.e. for $\sigma = 0.18 \text{ GeV}^2$, $\alpha'_L(L \gg 1) = 0.88 \text{ GeV}^{-2}$ is larger than for $L \leq 4$ and coincides with α'_J for the ρ -trajectory. Such a picture is partly seen in experiment, where for $L = 5$ the difference $M^2(a_6) - M^2(\rho_5)$ is relatively small and corresponds to the large value $\alpha'_L \approx 1.2 \text{ GeV}^{-2}$. However, this growth of α'_L is likely to be connected with another reason—an effective decreasing of the string tension at large distances due to new channels being opened. This effect will be considered in our next paper.

The calculated meson masses (see Table III) still are large compared to experiment and to get agreement between them a negative constant (a fitting parameter) must be added to the squared mass $M^2(nL)$ [5]. Here we shall not introduce a fitting constant, but instead take into account the quark self-energy correction to the meson mass.

VII. THE QUARK SELF-ENERGY CONTRIBUTION AND MESON MASSES

Recently it was observed that a negative constant must be added to the meson mass, which comes from the nonperturbative quark self-energy contribution created by the color magnetic moment of the quark [7]. This constant is rather large and was calculated with the use of the Feynman-Schwinger representation of the quark Green's function. The total nonperturbative self-energy contribution, both from the quark and the antiquark, was found to be fully determined by the string tension and by the current mass (flavor) of the quark:

$$\Delta_{\text{SE}}(nL) = -\frac{4\sigma\eta(f)}{\pi\mu_0(nL)}. \quad (7.1)$$

Here $\mu_0(nL)$ is just the constituent mass defined by Eq. (5.1). The constant $\eta(f)$ depends on the flavor: its numerical value for a quark of arbitrary flavor was calculated in Ref. [7], in particular for the light mesons we take as in Ref. [7]

$$\eta(n\bar{n}) = 0.90. \quad (7.2)$$

The self-energy terms, as well as the meson masses, are given in Table IV for the ground states ($n = 0, L \leq 5$) from which one can see that $\Delta_{\text{SE}}(L)$ decreases as a function of n and L , being proportional to $\mu_0^{-1}(nL)$. Still it is rather large (equal to -300 MeV) even for $L = 5$.

With the self-energy and the string corrections taken into account the spin-averaged meson mass $\bar{M}(nL)$ is fully determined. The Coulomb correction will be discussed in the next Section and calculated in Appendix B.

The meson mass is now given by

$$\bar{M}(nL) = M_0(nL) - \frac{\sigma\langle 1/r \rangle L(L+1)}{\mu_0(6\mu_0 + \langle \sigma r \rangle)} - \frac{4\sigma\eta}{\pi\mu_0}, \quad (L \leq 4) \quad (7.3)$$

and for the linear potential can be written as

$$\bar{M}(nL) = M_0(nL) - \frac{2\sigma\langle 1/r \rangle L(L+1)}{M_0^2} - \frac{16\sigma\eta}{\pi M_0}, \quad (7.4)$$

TABLE IV. The nonperturbative quark self-energy correction $\Delta_{\text{SE}}(L)$ and the meson masses $\bar{M}(L)$ in GeV for the ground states ($\sigma = 0.18 \text{ GeV}^2$, $\eta = 0.9$).

L	0	1	2	3	4	5
$\Delta_{\text{SE}}(L)$	-0.616	-0.460	-0.383	-0.335	-0.301	-0.294
$\bar{M}(0, L)$	0.723	1.279	1.685	2.017	2.30	2.514*
$\bar{M}_{\text{exp}}(0, L)$	0.612	$1a_J(1.252)$ $1f_J(1.245)$	$\pi_2(1.67)$ $\rho_3(1.69)$ $\omega_3(1.67)$	$a_4(2.014)$ $f_4(2.034)$	ρ_5 2.33 ± 0.04	$a_6(2.45 \pm 0.13)$ $f_6(2.47 \pm 0.050)$

*) this mass was calculated from the asymptotic formula (6.12).

using the relations (5.2) and (5.8). The calculated meson masses ($L \leq 4$) coincide with good accuracy with the experimental values (see Table IV).

For large L ($n = 0$)

$$\bar{M} = \sqrt{3\pi\sigma + 2\pi\sigma\sqrt{L(L+1)}} + \Delta_{\text{SE}}(L). \quad (7.5)$$

VIII. THE INTERCEPT OF THE REGGE TRAJECTORY

From the mass formula (7.4) it follows that the self-energy term enters $\bar{M}(nL)$ in such a way that the negative constant C_0 ,

$$C_0 = -\frac{32\sigma\eta}{\pi}, \quad (8.1)$$

appears in the squared spin-averaged mass $\bar{M}^2(L)$:

$$\bar{M}^2(nL) = (M_0 + \Delta_{\text{str}})^2 - \frac{32\sigma\eta}{\pi} + \left(\frac{16\sigma\eta}{\pi M_0}\right)^2. \quad (8.2)$$

Here the terms Δ_{str}^2 and $\Delta_{\text{str}}\Delta_{\text{SE}}$ will be neglected, because they give small contributions for $L \leq 4$, while the term Δ_{SE}^2 is kept, since it is not small in all states. The constant C_0 is rather large and for $\sigma = 0.18 \text{ GeV}^2$ is equal to -1.65 GeV . Using the expression (6.6) for the mass $(M_0 + \Delta_{\text{str}})^2$ and Eq. (6.1) for M_0^2 , Eq. (8.2) can be presented as

$$\bar{M}^2(L) = (\alpha'_L)^{-1} L + b(L), \quad (8.3)$$

with

$$b(L) = \sigma \left[3\pi - \frac{32\eta}{\pi} + \frac{32\eta^2}{\pi^2(L + 3\pi/8)} \right], \quad (L \neq 0),$$

$$b(L = 0) = \sigma \left[3\pi - \frac{32\eta}{\pi} + \frac{256\eta^2}{\pi^2 M_0^2(1S)} \right], \quad (8.4)$$

where for $M_0(L = 0)$ it is better to use the exact value, $M_0(1S) = 3.157\sqrt{\sigma}$ and α'_L was already defined by the expression (6.8). From (8.3) the intercept is

$$\alpha_L(0) = \alpha_L(M^2 = 0) = -\alpha'_L b(L = 0) = -\frac{b(L = 0)}{6.95 \sigma}. \quad (8.5)$$

Note that in $b(L)$ the combination $(3\pi - 32\eta/\pi)\sigma$ is a small number (equal 0.046 GeV^2 for $\sigma = 0.18 \text{ GeV}^2$) and therefore for the intercept the contribution of the self-energy term Δ_{SE}^2 is dominant.

From Eq. (8.4) it is clear that $b(L)$ is very sensitive to the value of the flavor factor η , which may introduce an uncertainty on the order of 5%.

With the use of the analytical expression (8.4) and the exact value of $M_0(L = 0)$, $\eta(n\bar{n}) = 0.90$ the quantity $b(L = 0)$ is equal to

$$b(L = 0) = 2.365 \sigma. \quad (8.6)$$

Then the intercept given by Eq. (8.5) takes the value

$$\alpha_L(0) = -(\alpha'_L) b(L = 0) = -2.356/6.95 = -0.34. \quad (8.7)$$

This number is in good agreement—larger by 10% only— with the experimental value $\alpha_L(0) = -0.30 \pm 0.02$. It is essential that the intercept does not depend on the string tension but instead is very sensitive to the flavor parameter η . Just for this reason the intercept for the mesons with different flavor depends on the flavor.

So, finally, the Regge L -trajectory calculated in the QCD string approach with $\sigma = 0.18 \text{ GeV}^2$ is fully determined,

$$L = 0.80 \bar{M}^2(L) - 0.34 \quad (8.8)$$

and appears to be very close to Eq. (2.3) obtained from a fit to the experimental spin-averaged meson masses, see Fig. 1. From Eq. (8.8) the averaged mass $\bar{M}(\pi - \rho)$ is found:

$$\bar{M}^2(1S) = 0.425 \text{ GeV}^2 \text{ or } \bar{M}(1S) = 0.652 \text{ GeV}, \quad (8.9)$$

which corresponds to a π -meson mass $\bar{M}(\pi) = 301 \text{ MeV}$. This number turns out to be smaller than $M(1S) = 0.723 \text{ GeV}$ calculated directly from Eq. (6.5) and this discrepancy illustrates how sensitive $\bar{M}(1S)$ is to the approximations used.

IX. COULOMB INTERACTION

In the previous sections good agreement of the spin-averaged meson masses (for the ground states with $L \neq 0$) was obtained without taking into account the Coulomb interaction. It is of interest to check whether the Coulomb effects are actually suppressed for $L \geq 0$ states and how large is Coulomb correction to $\bar{M}(\pi - \rho)$.

To this end we solve the Salpeter equation with the string potential taken as a linear plus Coulomb term, i.e., with the Cornell potential:

$$V_C(r) = \sigma r - \frac{4}{3} \frac{\alpha_s}{r}, \quad (9.1)$$

where $\alpha_s = \text{constant}$ can be used, since the light mesons have very large sizes, $R \geq 1.0 \text{ fm}$, and at such distances the strong coupling is saturated and close to the “freezing” value [8]. For α_s we take just the same value as for heavy quarkonia [14,15],

TABLE V. The spin-averaged masses $\bar{M}_C(L)$ in GeV, theoretical and experimental, for the ground states ($n=0$) ($\sigma = 0.19 \text{ GeV}^2$, $\alpha_s = 0.39$).

L	0	1	2	3	4	5
$\bar{M}_C(L)$	0.632	1.220	1.650	2.00	2.29	2.51
$\bar{M}_{\text{exp}}(L)$	0.612	$\bar{M}(f_J) = 1.24$	$\pi_2(1.66)$ $\rho_3(1.69)$ $\omega(1.65)$ $\omega_3(1.67)$	$a_4(2.014)$ $f_4(2.03)$	$\rho_3(2.30)$	$a_6(2.45)$ $f_6(2.47)$

$$\alpha_s = 0.39 \quad (9.2)$$

and for the string tension we use $\sigma = 0.19 \text{ GeV}^2$. However, the masses of the ground states, including the $1S$ state, can be nicely described with a smaller value for the coupling constant, $0.20 \leq \alpha_s \leq 0.39$, if correspondingly the value of σ is taken from the range $0.18 \text{ GeV}^2 < \sigma \leq 0.19 \text{ GeV}^2$. Therefore, the value $\alpha_s \approx 0.40$ can be considered as an upper limit for α_s compatible with the correct values of the meson masses. For larger values of α_s the masses of the $1S$ and $1P$ states will be too low.

The main characteristics of the $q\bar{q}$ system like the eigenvalues $M_C(nL)$ of Eq.(4.6) using the Cornell potential, the constituent masses $\mu_C(nL)$ defined by Eq. (5.1) together with the string and the self-energy corrections are presented in Appendix B in Tables X and XI. Here in Table V we give only the results of our calculations for the spin-averaged masses $\bar{M}_C(nL)$. Note that in the Coulomb case the relation (5.2) is not valid and therefore the meson mass $\bar{M}_C(nL)$ as well as Δ_{str} and Δ_{SE} should be written through the constituent mass (denoted as $\mu_C(nL)$) as in Eq. (7.3) (see Table X where the eigenvalues are given for $\sigma = 0.19 \text{ GeV}^2$, $\eta = 0.90$, and $\alpha_s = 0.39$).

With the use of the string and the self-energy corrections from Table XI the spin-averaged meson masses $\bar{M}_C(L)$, Eq. (7.3), are determined and their values are given in Table V together with the experimental numbers.

If now one compares the meson masses $\bar{M}_C(L)$ with those for the linear potential from Table III, then one can see that in the Coulomb case for the $1S$ and $1P$ states a better agreement with the experimental numbers is obtained, however, in the Coulomb case the string tension appears to be larger, $\sigma = 0.19 \text{ GeV}^2$. The calculated mass $\bar{M}_C(1S) = 0.632 \text{ GeV}$ (for the σr potential it is 0.732 GeV , $\sigma = 0.18 \text{ GeV}^2$) is very close to the value Eq. (8.9) from the Regge trajectory Eq. (8.7).

Now the Coulomb correction can be formally defined as the difference between the exact eigenvalues, $\bar{M}_C(L)$ and $\bar{M}(L)$:

$$E_C(\text{exact}) = \bar{M}_C(nL) - \bar{M}(nL) \quad (9.3)$$

and compared with the Coulomb corrections $E_C(\text{pert})$:

$$E_C(\text{pert}) = -\frac{4}{3}\alpha_s \langle 1/r \rangle \quad (9.4)$$

obtained when the Coulomb interaction is considered as a perturbation (see Table VI). In Eq. (9.4) the m.e. $\langle 1/r \rangle$ is to be taken for the linear potential with the same σ as in the Cornell potential.

TABLE VI. The exact and perturbative Coulomb corrections $E_C(L)$ (in MeV)

L	0	1	2	3	4	5
$E_C(\text{exact})$	-219	-132	- 103	- 86.7	- 76.2	- 68.8
$E_C(\text{pert})$	-194	-126	-99.4	- 84.9	- 75.1	- 68.1
difference	11.4%	4.5%	3.5%	2.0%	1.4%	1.0%

The numbers in Table VI demonstrate that the exact and perturbative corrections coincide with an accuracy better than 5% for all states with $L \geq 0$ (for the $1S$ state the difference is 11%) and therefore these corrections can be calculated as a perturbation.

For the nL states one should also take into account the difference between the exact constituent mass $\mu_C(L)$ and $\mu_0(L)$ for the linear potential; they are related as follows

$$\begin{aligned}\mu_C(1L) &\approx \mu_0(1L) + |E_C|/3 \quad (n=0), \\ \mu_C(nL) &\approx \mu_0(nL) + |E_C|/4, \quad (n \neq 0).\end{aligned}\tag{9.5}$$

This correction to the constituent mass is mostly important for the $1S$ state. For larger n the difference between μ_C and μ_0 can be neglected. As seen from Table VI, due to the Coulomb interaction all masses are shifted down by an amount in the range of 60 to 100 MeV and therefore a larger value of σ is needed, $\sigma = 0.19 \text{ GeV}^2$ for $\alpha_s = 0.39$, than for the linear potential.

However, one cannot take an arbitrary or too large value for σ , otherwise the Regge slope α'_L would be small and in contradiction with the experimental value. Therefore, in the Coulomb case only values $\sigma = 0.19 \pm 0.10 \text{ GeV}^2$ are allowed. Then to obtain agreement with experiment using $\sigma \leq 0.20 \text{ GeV}^2$ a restriction on the value of the strong coupling constant is found:

$$\alpha_s \leq 0.40 \quad (\sigma \leq 0.20 \text{ GeV}^2),\tag{9.6}$$

otherwise correct numbers for the Regge slope and the intercept cannot be obtained simultaneously.

This upper limit (9.6) for α_s appears to be in accord with the freezing value of the two-loop $\alpha_B(q^2 = 0) = 0.45$ (with the QCD constant $\Lambda^{(3)} = 330 \text{ MeV}$, $N_f = 3$) obtained in background field theory [7].

X. CONCLUSIONS

In the framework of the QCD string approach the spin-averaged meson masses with $L \leq 5$ ($n = 0$) have been calculated and expressed through a single parameter—the string tension σ and a set of universal numbers. In this approach the kinetic energy is of the same type as in the spinless Salpeter equation. The constituent mass and the nonperturbative quark self-energy are calculable and also depend on the string tension only.

This is the first time accurate predictions for the meson masses have been obtained relying on one parameter only, that is directly connected to the confinement mechanism in QCD.

TABLE VII. The eigenvalues $M_0(nS)$, constituent masses $\mu_0(nS)$, $\tilde{\mu}_0(nS)$, and matrix elements $\langle 1/r \rangle$ (in GeV) for the Salpeter Eq. (4.6) and the Airy equation Eq. (5.4) with the linear potential σr ($\sigma = 0.18 \text{ GeV}^2$, $L = 0$).

n	0	1	2	3	4
$M_0(nS)$	1.3394	1.9980	2.4985	2.9151	3.2797
$\mu_0(nS)$	0.3348	0.4995	0.6246	0.7289	0.8199
$\tilde{\mu}_0(nS)$	0.3519	0.5351	0.6703	0.7826	0.8807(3)
$\langle 1/r \rangle$ (SSE)	0.3638	0.3299	0.2959	0.2734	0.2559(5)
$\langle 1/r \rangle^a$ (EA)	0.3328	0.2669	0.2334	0.2118	0.1996(5)

^{a)} These m.e. are calculated from the Eq.(18) with $\tilde{\mu}_0(nS)$ defined by Eq.(20).

The analytical expressions for the slope and the intercept of the Regge L -trajectory (when the spin splittings are not taken into account) have been deduced, giving rise to a value $\alpha'_L = (6.95\sigma)^{-1} = 0.80 \text{ GeV}^{-2}$ ($\sigma = 0.18 \text{ GeV}^2$) which coincides with the experimental number. This L -trajectory can be considered as a universal one since in the approximation of closed channels it does not depend on spin and isospin.

It is shown that the Regge intercept does not depend on σ and $\alpha(M = 0)(\text{theory}) = -0.34$ turned out to be only 10% larger than $\alpha(M = 0)(\text{exp}) = -0.30 \pm 0.02$. From this intercept $\bar{M}(1S) = 652 \text{ MeV}$ corresponds to a π -meson mass equal to 300 MeV (chiral effects have been neglected here).

For all orbital excitations with $L \neq 0$ the calculated masses are in a good agreement with existing experimental data.

In order to obtain this good agreement with the data we find it necessary to impose a restriction on the value of α_s that is in accord with the freezing picture.

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APPENDIX A: DETAILED SPECTRA

The eigenvalues and the wave functions of the SSE equation were calculated with the help of the code used before [15,16]. The eigenvalues and relevant matrix elements are given in Tables VII and VIII for the linear potential and in Tables X-XI for the Cornell potential.

From Table VIII one can see that the difference between the m.e. $\langle 1/r^3 \rangle$ for the SSE and the Airy equations for the P-wave states turn out to be large reaching 40% for $n \geq 2$. As briefly discussed in Sect. III the reason behind these differences lies in the different asymptotic behaviors of the eigenfunctions of these two equations. In Table IX we also give the constituent masses Eq. (4.6) and the m.e. $\langle 1/r \rangle$ and $\langle 1/r^3 \rangle$ for the ground states ($n = 0$) with $L \leq 6$.

The calculated m.e. $\langle 1/r \rangle$ is used to obtain the string and Coulomb corrections, while the m.e. $\langle 1/r^3 \rangle$ can be used to calculate the hyperfine and fine-structure splittings for the mesons with $L \neq 0$.

TABLE VIII. The matrix elements $\langle 1/r^3 \rangle$ (GeV^{-3}), mass eigenvalues $M_0(nP)$ (GeV), and constituent masses $\mu_0(nP)$ and $\tilde{\mu}_0(nP)$ (GeV) for the P-wave states ($\sigma = 0.18 \text{ GeV}^2$).

n	0	1	2	3	4
$M_0(nP)$	1.7924	2.3153	2.7505	3.1291	3.4682
$\mu_0(nP)$	0.4481	0.5788	0.6876	0.7823	0.8671
$\tilde{\mu}_0(nP)$	0.4620	0.6115	0.7320	0.8335	0.9278
$\langle 1/r^3 \rangle$ (SSE)	0.0264	0.0422	0.0539	0.0635	0.0718
$\langle 1/r^3 \rangle$ (EA) = $\tilde{\mu}\sigma/4$	0.0208	0.0275	0.0329	0.0376	0.0417

^{a)} see footnote to Table VII.

TABLE IX. The constituent masses μ_0 (GeV) and the matrix elements $\langle 1/r^k \rangle$ (GeV^k), ($k=1,3$), of the SSE for the ground states ($n=0$) ($\sigma = 0.18 \text{ GeV}^2$, $L \leq 6$).

L	1	2	3	4	5	6
$\mu_0(0L)$	0.4481	0.5387	0.6163	0.6853	0.7480	0.8058
$\langle 1/r \rangle$	0.2362	0.1867	0.1589	0.1406	0.1274	0.1173
$\langle 1/\rho \rangle \sqrt{L+1}$	0.787	0.762	0.742	0.741	0.736	0.732
$\langle 1/r^3 \rangle$	0.0264	0.0098	0.0054	0.0035	0.0026	0.0019

From Table X one can see that in the Coulomb case the constituent masses $\mu_C(1S)$ and $\mu_C(1P)$ are larger by 29% and 10% respectively, than for the linear potential (see Table VIII) and therefore Δ_{SE} is smaller for them.

APPENDIX B: RESULTS FOR THE CORNELL POTENTIAL

In this appendix we present some auxiliary values for the Cornell potential.

TABLE X. The eigenvalues $M_C(L)$, the constituent masses $\mu_C(L)$ and the m.e. $\langle 1/r \rangle$ (in GeV) for the SSE with the Cornell potential for the ground states ($n = 0$). ($\sigma = 0.19 \text{ GeV}^2$, $\alpha_s = 0.39$)

L	0	1	2	3	4	5
$M_C(L)$	1.157	1.710	2.111	2.446	2.740	3.005
$\mu_C(L)$	0.415	0.496	0.580	0.656	0.710	0.745
$\langle 1/r \rangle_L$	0.484	0.266	0.202	0.170	0.145	0.130

TABLE XI. The string and self-energy corrections (in GeV) for the SSE with the Cornell potential ($\sigma = 0.19 \text{ GeV}^2$, $\alpha_s = 0.39$).

L	0	1	2	3	4	5
Δ_{str}	0	-0.051	-0.086	-0.112	-0.075	-0.068
Δ_{SE}	-0.525	-0.439	-0.375	-0.332	-0.307	-0.282

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